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# REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.



In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

***The Cambridge companion to Carnap***, edited by Michael Friedman and Richard Creath, Cambridge University Press, 2007, xvii + 371 pp.

The contemporary significance and remarkable depth of Rudolf Carnap's views continue to be elucidated and reaffirmed with this anthology. It clearly achieves the Cambridge companions' objective of providing rigorous yet accessible expositions of important philosophical work. Ten essays cover: Carnap's early philosophical development and relations with Frege, Husserl, Quine, Russell, and the larger Vienna Circle; his first views of geometry, rejection of metaphysics in the *Aufbau*, and sophisticated attempts to develop a probabilistic measure of inductive confirmation; and, the pragmatic character of his philosophy, especially the principle of tolerance. Four philosophically rich essays of most relevance to this journal consider: (i) Carnap's contributions to the development of modern logic [Erich Reck]; (ii) the coherence of his initial logicism about mathematics and later tolerance about foundational issues [Thomas Ricketts]; (iii) the search for a defensible, semantic concept of analyticity [Steve Awodey]; and, (iv) problems with equating the factual content of scientific theories with their Ramsey sentence [William Demopoulos].

(i) "Axiomatic Foundations of Kinematics" was Carnap's first dissertation project, scuttled by unsympathetic advisors, but later pursued under the rubric ***Untersuchungen zur allgemeinen Axiomatik*** (Wissenschaftliche Buchgesellschaft, posthumously published in 2000, edited by T. Bonk and J. Mosterin) and partially published as ***Abriss der Logistik*** (Springer Verlag, 1929), one of the first modern logic textbooks. Reck highlights this work's understudied


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and underappreciated contributions to modern logic, made in efforts to integrate Frege and Russell's views of logic with axiomatic approaches encouraged by Hilbert.

The issue was how to characterize axiomatic system completeness. Partly in correspondence with Carnap, Adolf Frankel distinguished three senses: deductive, semantic, and categorical (using contemporary terminology). Without a well-developed background logical theory, Frankel could not determine their interrelationships. Carnap had the innovative idea to formulate the issue in a higher-order logic with simple types from Frege and Russell, whose aversion to axiomatic methods prevented such a move. Within this framework, Carnap hoped to establish axiomatic systems are: (1) consistent iff satisfiable; (2) semantically complete iff categorical; (3) deductively iff semantically complete. Despite Hilbert's contagious optimism, Gödel's incompleteness theorems (IT) showed the 'only if' of (1) and (3) are false. Although these entailments hold for first-order logic, they fail for formal systems at least as strong as the fragment of Peano arithmetic called Robinson's arithmetic (see R. Smullyan, ***Gödel's Incompleteness Theorems***, Oxford University Press, 1991). But Reck locates the fundamental obstacle to Carnap's ambitions elsewhere, in his failure to distinguish syntactic from semantic consequence in notions of deducibility and logical consequence. By adopting Frege and Russell's conception of logic, Carnap assumed a fixed, "universal" background language that precludes the distinction between object and meta-language required to define syntactic and semantic consequence clearly. This shortcoming was apparently not discerned by several prominent logicians (Reck, p. 191) until Tarski finally did and convinced Carnap.

Regarding (2), categoricity implies semantic completeness, but Carnap's proof of the converse incorrectly assumed all models of higher order theories are definable. With this restriction it follows, but the general claim, labeled "Carnap's Conjecture," remains open. Usefully, Reck identifies other interesting projects Carnap's axiomatic work suggests, such as exploring his contention that concepts introduced by "complete" axiomatic systems (besides explicit definitions) have special significance to science, and the idea that truth in a mathematical theory's domain is representable by the logical consequences of its "complete" axiomatization. Gödel's IT might seem to make these projects largely pointless, but Ricketts and Awodey show otherwise.

(ii) Partly in response to IT, Carnap abandoned the "universalist" conception of logic and proposed the principle of tolerance (PT) in the ***Logical Syntax of Language***. After helpfully recounting its main themes—particularly that logical-mathematical sentences were represented as empirically-neutral (i.e., neutral with respect to observation sentences) logical consequences of transformation rules of formal axiomatic systems (called analytic or '$L$-determinate' sentences)—Ricketts asks about the status of underivable arithmetic truths IT guarantees exist for suitably strong systems. They are $L$-indeterminate, but presumably not empirically testable synthetic truths for Carnap. This apparently leaves only the non-analytic, non-empirical category of metaphysical pseudo-sentence. Carnap avoids this problematic conclusion by including the infinitary $\omega$-rule in his definition of consequence, which completes arithmetic and precludes such $L$-indeterminate truths. A basis for employing the stronger mathematics seems needed, however, to avoid justificatory infinite regress, a criticism Gödel made first (*Is Mathematics Syntax of Language?* in ***Collected Works: Unpublished Essays and Lectures***, vol. III, Oxford University Press, 1995, edited by S. Feferman et al., pp. 334–362). Ricketts explains this complaint is misplaced against Carnap. Carnap's objective is to formulate a syntactic criterion for analyticity delineating what are typically considered logical-mathematical sentences from empirical ones, not to give an account of mathematical truth *justifying* their analytic status. The latter might warrant restrictions on the mathematical resources used to construct analyticity criteria, but it runs counter to PT's insistence that philosophically foundational questions be abandoned for pragmatic ones about language use. Tolerance ensures Carnap is at liberty to utilize whatever resources are required to affect the desired analytic-synthetic distinction (see also W. Goldfarb and

T. Ricketts, *Carnap and the Philosophy of Mathematics*, in ***Wissenschaft und Subjektivität***, Academie Verlag, 2002, edited by D. Bell and W. Vossenkuhl, pp. 61–78).

Carnap's tolerance is not neutral. The analyticity criterion described above for this language (Language-II), which Carnap later recognized was equivalent to Tarski's definition of semantic truth, requires the $\omega$-rule so the underlying logicist motivation presupposes a strong meta-language possibly unpalatable to advocates of weaker logics. Ricketts argues that does not indicate an incompatibility between logicism and PT as some suggest (Neil Tennant gives a similar analysis in *Carnap, Gödel, and the Analyticity of Arithmetic*, ***Philosophia Mathematica***, vol. 16 (2008), pp. 100–112). Rather, it means the explication of the contentlessness of logical-mathematical sentences afforded by the stronger meta-languages may be unavailable to those advocates. This pragmatic weakness must be weighed against the merits of weaker meta-languages: exactly what PT counsels. After all, in full light of PT, Carnap also preferred languages for science incorporating classical mathematics.

(iii) Awodey takes up the analyticity project where Ricketts leaves off, in Carnap's post-***Syntax*** "semantic" works. Carnap accepted Tarski's semantics, which posed problems because Tarski's truth definition does not distinguish logical from semantic truth in languages containing logical and non-logical terms. But only these languages can be adequate for science in general. One solution the ***Syntax*** considered is isolating logical from non-logical terms by $L$-determinacy: logical terms are the largest set of terms such that sentences containing only them are $L$-determinate. True sentences by Tarski's definition containing only logical terms could then be understood to delimit logical truth. This procedure is independent of interpretations of non-logical terms and would apply across different interpretations and languages.

The mathematician Saunders Mac Lane showed this method was deficient—no unique largest set of such symbols exists (*Carnap on Logical Syntax*, ***Bulletin of the American Mathematical Society***, 1938, pp. 171–176)—and Carnap's successive attempts in ***Foundations of Logic and Mathematics, Introduction to Semantics***, and ***Meaning and Necessity*** to pinpoint the distinction Awodey recounts were similarly fruitless, primarily because the semantic focus provided no resources for resolving the difficulty for Carnap. Even if it had, Awodey notes that whereas the modern view is that logical truth must be gauged against every possible interpretation of non-logical terms over every possible quantification domain, Carnap's semantic systems assumed a single fixed interpretation. In fact, this catholic, modern view was absent from logic throughout the 1940s. Ironically, although Carnap's efforts were unsuccessful, seeds of the modern view were present in his early axiomatic work (Awodey, p. 238) and were later successfully pursued by others (e.g., Tarski and Kripke). In particular, Tarski later built on the ideas motivating Carnap to identify a logical-descriptive distinction for terms.

(iv) After recounting influences from Frege and Russell's logic, Hilbert's axiomatization of geometry, and particularly Einstein's analysis of simultaneity, Demopoulos carefully describes Carnap's two-stage reconstruction or explication of scientific theories. The first distinguishes observational $O$-vocabulary from theoretical $T$-vocabulary. Demopoulous grants Putnam's criticism (*What theories are not*, in ***Logic, Methodology, and Philosophy of Science***, Stanford University Press, 1962, edited by E. Nagel, P. Suppes, and A. Tarski, pp. 215–227) that actual scientific terms and sentences cannot easily be so classified, but correctly notes its irrelevance. The relatively unproblematic distinction between observed and unobservable events, properties, and relations can be used to *impose* the desired distinction in vocabulary. The distinction was also conventional for Carnap, so a uniquely defensible division is unnecessary. Besides $O$ and $T$-sentences containing only their respective terms, correspondence $C$-rules contain both, thereby establishing connections between the theoretical domain of unobservables and the observables that provide the evidentiary basis for claims about the former. $C$-rules are therefore principles of empirical and/or epistemic interpretation without which logical empiricists would not agree $T$-sentences are semantically interpretable. Scientific theories are conjunctions of $T$-sentences and $C$-rules, $TC(O_1, \ldots, O_m; T_1, \ldots, T_n)$, where $O_i$ and $T_i$ represent terms.

The second stage produces the Ramsey sentence of $TC$:

$$(\exists X_1, \ldots, \exists X_1)TC(O_1, \ldots, O_m; X_1, \ldots, X_n), \qquad (R(TC))$$

*i.e.*, the existential quantification over the theoretical terms of $TC$. The Carnap sentence is then constructed:

$$R(TC) \rightarrow TC. \qquad (C(TC))$$

For Carnap, $R(TC)$ and $C(TC)$ represent the factual and analytic components of theories for three reasons:

(i) $R(TC) \wedge C(TC)$ is logically equivalent to $TC$;
(ii) $R(TC)$ and $TC$ imply the same $O$-sentences (which Carnap himself proved);
(iii) $C(TC)$ is observationally uninformative: all $O$-sentences it implies are logically true.

Demopoulos also observes that this factual-analytic division is independent of the formalization, and he shows it subsumes Carnap's earlier analysis of dispositional terms in *Testability and Meaning* (***Philosophy of Science***, vol. 3 and 4, pp. 419–471 and 1–40).

Despite these advantages, Demopoulos shows that a deficiency similar to one facing Russell's structural realism (***The Analysis of Matter***, Allen and Unwin, 1927) also plagues Carnap's explication: it does not capture how scientific theories are a posteriori and synthetic (see also Stathis Psillos, *Carnap, the Ramsey sentence, and realistic empiricism*, ***Erkenntnis***, vol. 52 (2000), pp. 253–279). The problem is that, whereas $TC$ represents *synthetic* theoretical claims, $R(TC)$ seems unable to do so. Given a minimal cardinality assumption about domains of quantification (and that the relevant set of $O$-sentences are consistent), $R(TC)$ makes such claims logically true and knowable a priori (an appendix presents the requisite proofs). To illustrate, consider a $TC$ comprised only of $(\forall x)[(Fx \rightarrow Rx) \wedge (Rx \rightarrow Kx)]$ where $F$ and $K$ are $O$-terms and $R$ is a $T$-term. Its observational content is that every $F$ is a $K$, but $TC$ also states that $R$ weakly separates them. The corresponding Ramsey sentence, however, which existentially quantifies over $R$, is (almost) logically true (modulo the cardinality and consistency assumptions). As a reconstruction of the claim $TC$ makes, this seems to mischaracterize its theoretical status.

Unfortunately, only the main points of these philosophically stimulating essays have been considered, but their detailed and careful analyses make it abundantly clear that Carnap's work demonstrates logical abilities beyond mere facility with application. They also reveal the philosophical insights and challenging issues that emerge from his unique, technically rigorous approach to philosophical issues. Hopefully, similar high quality works will continue to secure Carnap's rightful place as an architect and exemplar of analytic philosophy.

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***The Princeton Companion to Mathematics***, edited by Timothy Gowers (June Barrow-Green and Imre Leader, associate editors), Princeton University Press, 2008, 1008 pp.

The publishing event of the year in mathematics is without doubt ***The Princeton Companion to Mathematics***. Edited by Timothy Gowers, it is a thousand page (or so) encyclopedia of all things mathematical, from its theorems, to its history, to biographies of its principals, to its sociology ... to something of its philosophy.

This review will contain a word about the latter; but first, about the ***Companion***: Importantly, the quality of the exposition is very high. The entries, from those on "dimension" and "duality"—two of the volume's gems—to the more technical, are simply beautifully written. As to the content, this is also presented with a great deal of care. The reader, upon finishing an entry, understands *why* a particular mathematical concept should have been introduced; *why* a particular question was asked—the "why" of mathematics, as Michael Harris calls it. One often hears mathematicians bemoaning the present day state of mathematical exposition. The volume is not only a powerful corrective to what has been an unfortunate trend downward (in some fields); overall it leaves one with a sense of mathematics as a subject of great beauty and many depths, free of arbitrariness and full of meaning. It is a splendid piece of work.

As for logic and foundations, the book takes an interesting attitude toward them. It begins, surprisingly, with a passage from Bertrand Russell's ***The Principles of Mathematics***:

> Pure Mathematics is the class of all propositions of the form $p$ implies $q$, where $p$ and $q$ are propositions containing one or more variables, the same in the two propositions, and neither $p$ nor $q$ contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of *such that*, and any such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics *uses* a notion which is not a constituent of the propositions which it considers, namely the notion of truth.

About this passage the editor remarks "The ***Princeton Companion to Mathematics*** is about everything that Russell's definition leaves out."[1] Russell is saying something important here; he is pointing to the distinction between syntax and semantics at a time when the distinction was far from clear. Gowers goes on to say that while in 1903, the year the ***Principles*** was published, many mathematicians were interested in logical foundations, today's mathematician is "likely to have other concerns." And this is of course true, and was perhaps true even then. As the American philosopher Charles Sanders Peirce wrote already in 1903:

> It does not seem to me that mathematics depends in any way upon logic. It reasons, of course. But if the mathematician ever hesitates or errs in his reasoning, logic cannot come to his aid. He would be far more liable to commit similar as well as other errors there. On the contrary, I am persuaded that logic cannot possibly attain the solution of its problems without greater use of mathematics. Indeed all formal logic is merely mathematics applied to logic.[2]

That said, there is a generous amount of space given to mathematical logic. Model theory, set theory and computational complexity are represented by three beautiful entries by Dave Marker and Joan Bagaria (both treated below), and Oded Goldreich and Avi Wigderson respectively; and chapter III, entitled "Mathematical Concepts," contains much in the way of content from mathematical logic. There is even an entry on the axiom of determinacy in chapter III, right up there with "metric spaces," "manifolds," and "Ricci flow."

Joan Bagaria's entry on set theory manages to cover all the main trends in this very technical topic. More than half of the entry consists of a careful introduction, up to Gödel's inner model of constructible sets, for the lay mathematician. He then dives into forcing; and although some have found the topic intimidating, Bagaria manages to give a gentle, formula-free account of it. The second half of the entry covers large cardinals, cardinal arithmetic, determinacy, descriptive set theory, and forcing axioms, ending with some remarks on Hugh Woodin's $\Omega$-logic. The essay is an excellent overview for any logician, and should take the

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[1] p. ix.

[2] "The Essence of Mathematics" in J. R. Newman (ed.) The World of Mathematics, New York: Simon and Schuster, 1956.

lay mathematician very far toward understanding what set theorists are up to, in trying to find order not only in sets of reals, but even in the wilderness of completely arbitrary sets.

David Marker's "Logic and Model Theory" gives an excellent introduction to mathematical logic. He covers Gödel's Completeness and Incompleteness Theorems and discusses their consequences, most notably the undecidability of the theory of the natural numbers. He then covers the other side of things, namely the fact that the theories of the reals and complex numbers are decidable, explaining the important and close connections to algebraic and analytic geometry along the way. This entry, especially when combined with Bagaria's on on set theory, provides the lay mathematician with a lucid introduction to the main concepts and results of mathematical logic.

On the side of foundations, most of chapter II is devoted to the history of the notions of rigor and proof, in very informative entries written by Tom Archibold and Leo Corry respectively; and the foundational debate of the 1920s, particularly the story of the debate over non-constructive methods, is ably recounted in José Ferreirós's too brief entry. (Though his assertion there that "formalism established itself in practice as the avowed ideology of twentieth century mathematicians"[3] certainly contradicts this reader's experience.)

As for what is omitted, probably every mathematician could name a treasured theorem left out of Chapter V's list of theorems and problems. And though comprehensiveness is surely unattainable here, for what it's worth, this reviewer's vote would have gone to a good presentation of Saharon Shelah's Main Gap, a powerful theorem which divides mathematical theories into the so-called structure and non-structure cases; a realization of the Hilbert program of a sort, and the "conclusion of a line of research which had cost him fourteen years of intensive work and not far off a hundred published books and papers."[4] The only other remark one perhaps might make along these lines is that Julia Robinson's biography could have been included among the 96 biographies given in chapter VI.

The Peirce point of view does come through in the selection of what's covered. One would have hoped that the early debates on constructive proofs, for example, as detailed in Ferreirós's entry, would have inspired some reflection on foundational issues along contemporary lines—even so that this is not *necessarily* part of mathematics proper. It is actually interesting that such are almost completely absent from the volume; but perhaps it is best left to the readers of this ***Bulletin***, to reflect on the reason why this is.

Two entries do stand well within the contemporary philosophical culture. John Burgess's "Analysis, Mathematical and Philosophical" in the history as philosophy genre, so-called, appears in the chapter on the influence of mathematics in various fields. And Michael Harris's "Why Mathematics? You Might Ask" occupies the territory of, roughly, post-modernism and appears in the chapter called "Final Perspectives." It is a tribute to the editors that these two papers should exist in the same collection; in fact reading them back to back is an interesting experience.

Burgess's entry traces the evolution of exact methods in philosophy in the context of the rationalist/empiricist dichotomy, with "rationalists the party of reason, and empiricists the party of experience."[5] The entry is about the influence of mathematics on philosophy, and accordingly Burgess begins with Descartes and Leibniz, "impressed by the apparent ability of pure thought ... to achieve, as it seemed to do in geometry, substantive results with worldly applications."[6] But how are such achievements possible? As Burgess observes,

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[3]p. 155.

[4]as described by Wilfrid Hodges in "What is a Structure Theory," Bulletin of the London Mathematical Society, (1987), vol. 19, 209–237. Shelah's abstract announcing the final step of the proof was called "Why am I so happy?", Abstracts Amer. Math. Soc., 3 (1982), 282.

[5]p. 928.

[6]Ibid.

"... for rationalists mathematics was a source of *methods*, for empiricists it was the source of a *problem*."[7] Burgess takes the reader through Kant's view of the problem before passing to the rise of analytic philosophy, in which the founding fathers are considered in the light of the influence of mathematics on their thought. For Russell and Frege, the influence of mathematics was decisive, in both the direct and indirect senses; for subsequent generations of analytic philosophers the influence is indirect, though massive nevertheless, inheriting the Frege/Russell tradition as they do.

In the second part of the entry Burgess rehearses the movement toward rigorization and generalization which marked mathematical analysis in the nineteenth century. Of crucial importance for analytic philosophy was the generalization of the notion of "function" due to Cauchy, Riemann and others, wherein the requirement that a function be given by an explicit formula was dropped, along with the requirement that a function have the reals as domain and range. For Frege the innovation was utterly decisive, and led directly to his formulation of first order *relational* logic. As Burgess quotes Frege on the generalization of the notion of "function" in his paper "Function and Concept": "In both directions I will go further."[8]

The third part of the entry deals with another important episode in the history of analytic philosophy as influenced by mathematical analysis, namely Russell's Theory of Definite Descriptions. Burgess contends that " ... the method of contextual definition, which the theory of descriptions exemplifies, was inspired by the nineteenth century rigorization of analysis,"[9] a point, Burgess observes, which is perhaps underappreciated, even by historians. (A contextual definition, as Burgess defines it, is " ... a definition that does not provide an analysis of a word or phrase standing alone, but rather provides an analysis of contexts in which it appears."[10]) Indeed at least this reviewer has never encountered the connection.

Burgess, characteristically, takes the long view. In the middle of his ruminations on Frege he pauses to make this remark, on the nature of philosophy itself:

> Indeed, to a large degree philosophical analysis simply *is* the logical analysis of philosophical rather than mathematical notions, carried out with the aid of Frege's broad new logic, or still broader extensions of it introduced by his successors.[11]

Michael Harris's "Why Mathematics?" is surely among the most astonishing entries in the volume. The entry, about the notion of "metaphysical certainty," from the post-modern perspective and from his own, evinces a somewhat dismissive attitude toward the foundations and philosophy of mathematics in the analytic tradition—though one has to qualify such a judgement in the case of Harris, a reputable number theorist whose philosophical writings[12] can be very subtle.

That said, Harris's view on foundations of mathematics seems to be something like the following: The notion of certainty, mathematical or otherwise, has become "philosophically discredited."[13] To the extent that the philosophy of mathematics is concerned with the notion—and it is—or with the development of top-down "master narratives," or with the notion of truth simpliciter, it is doomed to irrelevance:

> ... those with tendencies that I have described as postmodernist continue to express skepticism regarding certainty, *seemingly unaware that their target is now*

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[7] Ibid.

[8] p. 931.

[9] p. 933.

[10] p. 934.

[11] p. 931.

[12] Harris has written about 5 philosophical papers out of a total of 70. The remainder are mostly in the area of number theory.

[13] p. 973.

> *little more than an advertising slogan that has little to do with the real concerns of mathematicians . . .* [14]

and failure:

> The struggle to . . . find out the "essence" of mathematics is why the philosophy of mathematics keeps visiting the scenes of its many past defeats.[15]

Wider concerns are dealt with thus:

> By far the larger part of activity in what goes by the name of philosophy of mathematics is dead to what mathematicians think and have thought . . . [16]

In sum,

> . . . the most striking single feature of [twentieth century philosophy of mathematics] is that it is very largely banal.[17]

"Bridging the gap between mathematicians and metaphysicians is probably hopeless,"[18] Harris writes. And as for meaning, Harris quotes Roland Barthes: "Meaning is what makes things sell"[19]—a pretty jaundiced view of the matter, all things considered.

Harris's remarks are not without merit. But the case against analytic philosophy of mathematics Harris offers here, based as it is on a very selective reading of the literature, is somewhat flawed. For example, though he admits to making only a random walk through that literature,[20] with all due respect to those he does cite, he has managed to avoid virtually every single classic of the analytic philosophy of mathematics genre—a priorist or otherwise. There is also the entanglement with postmodern philosophy. Harris is clearly interested in it; but there is no question that the level of clarity which characterizes the analytic tradition in philosophy of mathematics at its best, as exemplified, e.g., by John Burgess's contribution to this very volume, has simply not been seen in the post-modern literature on mathematics, to say the least—whatever its virtues.

Harris's entry may give the appearance of a mere critique of the status quo in philosophy of mathematics, but it is more than that.[21] What warrants its inclusion in the ***Companion*** is its very well-observed positive part, in which, in a vein somewhat reminiscent of the writings of Gian-Carlo Rota, Harris outlines his own naturalistic agenda. Naturalism, once again prevalent in the philosophy of mathematics literature, is usually attributed to Quine, another opponent of the a priorist conception of philosophy, who formulated the view thus:

> . . . [naturalism] is the recognition that it is within science itself, and not in some prior philosophy, that reality is to be identified and described.[22]

But the flavor of Harris's naturalism is very different. It is less a fully fledged position in the empiricist tradition, as was Quine's, than a recommendation to spend one's philosophical coin on the "why" of mathematics: on distinctions such as those between "illuminating concepts" and "confirming theorems,"[23] very much deemphasizing the latter, an on such

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[14] p. 970. Italics the reviewer's.

[15] p. 969.

[16] p. 971. Harris is quoting from David Corfield's ***Towards a Philosophy of Real Mathematics***, Oxford 2003.

[17] p. 971. Harris is quoting from Ian Hacking's ***Historical Ontology***, Harvard, 2002.

[18] p. 971.

[19] p. 966.

[20] p. 972.

[21] Many of the above-cited remarks, as well as those of a similar critical nature, are a somewhat disparate collection of citations from the literature. It is therefore perhaps not entirely clear what the specifics of Harris's own criticism amounts to.

[22] p. 21, ***Theories and Things***, Harvard 1981.

[23] p. 973.

value-laden terms as "naturality" and "structure." The mathematician's own reflections deserve consideration, and more, the conversation in philosophy of mathematics should be itself mathematically rich.

The idea of naturalizing mathematics is very promising. The idea has also surfaced in other quarters, as was mentioned. But Harris's suggestion to naturalize mathematics comes from within the subject. He challenges philosophers to take, for example, Gowers's remarks on the nature of mathematics, occurring on the very first page of the preface to this massive volume, seriously; to look at what mathematicians actually do; to become engaged with their reflections on the nature of the strange and magnificent enterprise that mathematics has become—to read, in short, this book.

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ULRICH BLAU. ***Die Logik der Unbestimmtheiten und Paradoxien.*** Philosophische Impulse, vol. 8. Synchron Wissenschaftsverlag der Autoren, Heidelberg, 2008, 960 pp.

Soon after I had come to Munich in 1986 to study philosophy I heard a rumor that one of the professors there was working on a huge manuscript attacking all the paradoxes of logic which unfortunately was supposed to be too technical for a philosopher's digestion. In 1990, I finally got hold of a draft of this manuscript, and I started taking courses with its author, Ulrich Blau. These courses comprised the most impressive experience of my undergraduate career; Ulrich is an amazingly thorough, unorthodox, and deep thinker, an exhilarant platonist, and a hilarious controverter. At all times he has appeared to me as the incarnation of Husserl's slogan "Zu den Sachen selbst!" ("To the things themselves!") by being one of those rare examples of German philosophers who concentrate on discussing substantial issues rather than discussing other people's discussions. In his case the issues are: the logical analysis of natural language (with a stress on vagueness, intention, and quotation), the semantic and epistemic paradoxes (the liar sentence, the hangman paradox), and a set and class theoretical logic of reflection (with an "ultimate" extension of the field of real numbers into the transfinite and the infinitesimal). His approaches are always original and his solutions well-motivated, strong, and powerful. In his world view, the liar paradox, mathematical platonism, the formally unexpressible, and Nāgārjuna are tightly connected with each other. The Blau manuscript kept growing over the years, and it now finally appeared in print as ***Die Logik der Unbestimmtheiten und Paradoxien***. Fluent understanding of German is a prerequisite for being able to read this book, but a knowledge of logic is not, as it has a self-contained chapter on first order logic, called **L** here. There are people with more aggressive marketing strategies for their products than Ulrich is willing to use, so I can only hope that this will be another instance where "it is the stillest words that bring on the storm."

The current book includes the results of Ulrich Blau's previous one, ***Die dreiwertige Logik der Sprache***, de Gruyter, Berlin and New York, 1978, X + 275 pp., in which it is shown that the analysis of the phenomena of vagueness and non-referentiality of natural language produce a system **LN** of first order logic expanding **L** with three truth values *true*, *false*, and *indefinite* ("unbestimmt") and two forms of negation (corresponding to the two interpretations of "Pizarro did *not* find Eldorado" depending on whether we assume Eldorado to exist or not to exist). The new book also develops a logic **LQ** of quotation which expands **L** and which has quotation marks and two different kinds of variables, *de re* and *de dicto*, to range over objects and expressions, respectively. The system **LQ** is shown to contain arithmetic and is subject to the incompleteness phenomena. It is the most convincing and powerful logic of quotation I know to have appeared in the literature.

But I would like to have my review of ***Die Logik der Unbestimmtheiten und Paradoxien*** focus on his proposed solution of the semantic paradoxes and the set and class theories which get involved. A paradox appears if intuitive reasoning keeps entailing conclusions which contradict what is proven by a formal system. Blau's key idea is to allow the presence of a truth predicate in a formal language with arithmetic at the cost of making the process of reflection along an absolutely indefinite well-ordering $\Omega^*$ part of the formal system. His approach, first announced in his *Die Logik der Unbestimmtheiten und Paradoxien (Kurzfassung)*, ***Erkenntnis***, vol. 22 (1985), pp. 369–459, is somewhat reminiscent of "revision theories" of truth, but it is also much more advanced and superior to them, as none of the systems of revision theories which are in stock tries to catch up with the expressive power of our intuitive semantic reasoning about that system (like the one producing the revision sequences from attempts to find the extension of the truth predicate); but this is exactly what Blau works out, and he has answers where all of the revision theories rest in silence.

The liar paradox is given by the sentence $\varphi =$ "This sentence is not true." We understand this sentence, and we reason that if it is false or just "meaningless," then it is true, and that if it is true, then it is false. This very process of verification, Blau says, assigns the truth value *open* ("offen" $\neq$ indefinite) to $\varphi$ at the lowest level of reflection. Therefore, at the next level of reflection we verified $\varphi$ to be *true*, and at the second next level of reflection we verified $\varphi$ to be *false*, etc. If $\varphi$ is true/false at the $\alpha^{\text{th}}$ level of reflection, then $\varphi$ is false/true at the $\alpha + 1^{\text{st}}$ level of reflection. Once we passed through all the finite levels of reflection or more generally through all the levels below some limit ordinal $\lambda$, we verified $\varphi$ to be open at the $\omega^{\text{th}}$ or at the $\lambda^{\text{th}}$ level of reflection, respectively. Blau argues convincingly (cf. p. 451) that other proposed solutions to the semantic paradoxes lead into dead ends. His solution is a *tour de force* with a big amount of persuasive power. If this idea is to be integrated into a formal system, though, then the expressive power of the meta-language should be made available to the object language as well so that the latter may express statements like "At stage $\alpha$, the sentence $\psi$ is true." By the necessary well-foundedness of the verification process, at a given level $\alpha$ of reflection we only have access to the truth values of statements at earlier levels $\beta < \alpha$ and not to truth values that are only to be decided at higher levels $\beta > \alpha$. But doesn't the liar paradox then resurrect in new disguise? What about the sentence $\varphi^* =$ "This sentence is open at *all* levels of reflection"? It is indeed open at all levels of reflection, but therefore, as we reason intuitively, it is true. The subtety here, though, is given by the word "all" in $\varphi^*$. We need to take a closer look at Blau's logic of reflection and the "ordinals" $\alpha$ which index levels of reflection and which are provided by set and class theories.

Three logics of reflection are presented in ***Die Logik der Unbestimmtheiten und Paradoxien***, namely **LR**, **MR**, and **KR**. The system **LR** expands both **LN** and **LQ**. The role of having quotation available will be to have a device for coding integers at hand, and the overall idea behind generalizing both **LN** and **LQ** is to arrive at a system which analyzes the entire natural language. In addition to (, ), sentential connectives, quantifiers, variables *de re* and *de dicto*, function and relation symbols, **LR** has an indicator $\star$ (standing for the current level of reflection) and a symbol $T^\tau$ (for "is true at level $\tau$") for each term $\tau$ or for $\tau$ being equal to $\star$. As arithmetic can be modelled already in **LQ**, a term $\tau$ may end up as being interpreted by an integer $n \in \mathbb{N}$ in which case $T^\tau$ will get its obvious meaning; $T^\star$ will mean "is true at the current level." (Formally, $T^\tau$ will be allowed as a unary sentential connective as well as a predicate.) As self-referentiality is present already in **LQ**, the system **LR** can formulate a sentence $\varphi$ which is equivalent to $\neg T^\star\varphi$, i.e., **LR** can formulate the liar sentence. The logic **LR** has six truth values: *true*, *false*, *indefinite*, *open*, $\bar{\text{T}}$ (not true, but open if false or indefinite), and $\bar{\text{F}}$ (not false, but open if true or indefinite), and the set of levels of reflection is identical with $\omega = \mathbb{N}$. The semantics is defined in a natural way through verification rules. $T^\star\psi$ is true at level $n + 1$ iff $\psi$ is true at level $n$. An attempt to verify, falsify, or identify $\psi$ as indefinite at a given level of reflection need not produce an answer in which case $\psi$ will end up as being open, $\bar{\text{T}}$, or $\bar{\text{F}}$ at that level. The **LR**-version of the sentence $\varphi^*$ discussed above can be written as a sentence $\varphi^*_{\boldsymbol{LR}}$ which is equivalent to $\forall n \in \mathbb{N}\neg O^n\varphi^*_{\boldsymbol{LR}}$. (Here, $O$, standing for "open," is definable in terms of $T$.) Whereas the liar paradox might be considered as "solved," this sentence $\varphi^*_{LR}$ produces a new paradox: $\varphi^*_{LR}$ is open at all levels $n$, and therefore it is intuitively true. Another new paradox is given by the simpler sentence $\forall n \in \mathbb{N}\ T^n 2+2=4$, which is open at all levels of **LR**, but $2+2=4$ is true at all levels, whence $\forall n \in \mathbb{N}\ T^n 2+2=4$ is also intuitively true. These paradoxes are to be resolved by introducing more levels of reflection: $\omega$ and beyond.

This is where the line of thought is forced to bring set and class theory into play. Blau develops set theory and a hierarchy of class theories $\mathcal{M}'^{\alpha}_{\mathbb{V}}$. $\mathcal{M}'^{0}_{\mathbb{V}}$ is set theory. For any $\alpha$, $\mathcal{M}'^{\alpha}_{\mathbb{V}}$ results from $\mathcal{M}'^{\beta}_{\mathbb{V}}$, $\beta < \alpha$, by adding variables ranging over all classes which are definable in one the languages of $\mathcal{M}'^{\beta}_{\mathbb{V}}$, $\beta < \alpha$. Blau shows that $\mathcal{M}'^{\alpha}_{\mathbb{V}}$ is intertranslatable with $\mathcal{M}^{\alpha}_{\mathbb{V}}$, where for any $\alpha$, $\mathcal{M}^{\alpha}_{\mathbb{V}}$ results from $\mathcal{M}^{\beta}_{\mathbb{V}}$, $\beta < \alpha$, by adding the truth predicate for the languages $\mathcal{M}'^{\beta}_{\mathbb{V}}$, $\beta < \alpha$ (and $\mathcal{M}^{0}_{\mathbb{V}} = \mathcal{M}'^{0}_{\mathbb{V}}$). He also discusses impredicative classes and the relevant philosophical issues of platonism with respect to sets and (predicative or impredicative) classes and of relativism vs. absolutism. Almost *en passant* he also presents a platonistic argument in favor of CH, the continuum hypothesis. (Cf. also his *Ein platonistisches Argument für Cantors Kontinuumshypothese*, ***Dialectica***, vol. 52 (1998), pp. 175–202.)

Now following Blau's notation, let $\Omega$ denote the class of all ordinals, and let $\Omega^\bullet$ denote the length of the least class-sized well-ordering which has $\Omega$ as a strict initial segment and which is closed under "ordinal" addition and multiplication. The system **MR** comes from **LR** (basically) by adding a constant $\overline{\alpha}$ for every $\alpha \in \Omega^\bullet$, and the system **KR** comes from **MR** by adding set and class theoretical truth predicates. The class of levels of reflection now becomes identical with $\Omega^\bullet$. The semantics is again defined in a natural way through verification rules. If $\lambda$ is a limit, then $T^\star\psi$ is true at level $\lambda$ if $\psi$ is true at all but boundedly many levels $\alpha < \lambda$. The paradox given by $\varphi^*_{LR}$ is then "solved" (the sentence will become true from level $\omega$ onward), but the new variant of $\varphi^*$ is a sentence $\varphi^*_{LM/KR}$ which is equivalent to $\forall\alpha \in \Omega^\bullet\ \neg O^\alpha \varphi^*_{LM/KR}$ and which produces a new paradox: $\varphi^*_{MR/KR}$ is indeed open at all levels $\alpha$, and therefore it is intuitively true.

Blau goes on to argue that in order to resolve the paradox given by $\varphi^*_{LM/KR}$, one needs an ultimate logic of reflection which expands say **MR** in that it does not come with a *fixed* class of levels of relection (like $\omega$ in the case of **LR** and $\Omega^\bullet$ in the case of **MR**) but with an *open totality* $\Omega^*$ of all levels of reflection which is "extendable indefinitely" ("unbegrenzt fortsetzbar"): the apparent paradox provided by $\varphi^*$ will disappear if and only if we realize that the quantifier "for *all* levels of reflection ..." is as absolutely indefinite and formally incomprehensible as *the* longest (class sized) well ordering (cf. Thesis 15 on p. 132 and the discussion on p. 756). This move of ontological speculation in the light of the liar paradox constitutes the platonist's counterpart to the constructivist's conception of the reals as an "open totality" in the light of Cantor's diagonal argument. The progression $\Omega^*$, for Blau, yields a shortcut from mathematics to a mysticism that is "as clear as day." Even though $\Omega^*$ is formally unexpressible, Blau develops a "transreal" number theory based on it.

***Die Logik der Unbestimmtheiten und Paradoxien*** could be one of the most important contributions to the theory of semantic paradoxes of all times. It is a quarry of philosophical insights, of convincing results concerning the logical analysis of natural language, a compendium of interesting logical systems, a forceful attack to solve the semantic paradoxes, and, finally, it is also a beautiful piece of literature (I keep enjoying reading its wonderful dialogues!). It is to be hoped that Ulrich Blau will eventually receive the amount of attention that he truly deserves, and his proposals are overripe for an in-depth discussion in the logico-philosophical community.

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